4.1 - Preliminary Theory–Linear Equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (1)$$

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (2)$$

Are n^{th}-order, linear differential equations
(1) is homogeneous and if
$g(x) \not\equiv 0$, then (2) is nonhomogeneous.

$y(x_0) = y_0$, $y'(x_0) = y_1$, … $y^{(n-1)}(x_0) = y_{n-1}$
are initial conditions. An initial-value problem is (1) or (2), together with initial conditions.
In the event that we have information such as $y(a) = y_0$, $y(b) = y_1$, then we have a boundary-value problem.

Thm: Existence of a unique solution

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

These two important criteria will be assumed henceforth.

A boundary-value problem can have zero, one, or infinitely many solutions

Ex: The given two-parameter family is a solution of the indicated differential equation on the interval $(-\infty, \infty)$. Determine whether a member of the family can be found that satisfies the boundary conditions.

$y = c_1x^2 + c_2x^4 + 3; \quad x^2y'' - 5xy' + 8y = 24$

(a) $y(-1) = 0, \quad y(1) = 4$

(b) $y(0) = 1, \quad y(1) = 2$

(c) $y(0) = 3, \quad y(1) = 0$

(d) $y(1) = 3, \quad y(2) = 15$

a) $0 = c_1(-1)^2 + c_2(-1)^4 + 3 \Rightarrow 0 = c_1 + c_2 + 3$

$4 = c_1 + c_2 + 3$ No

b) $y(0) = 1 \Rightarrow 1 = 3$ No

c) $y(0) = 3 \Rightarrow 3 = 3 \checkmark \quad y(1) = 0 \Rightarrow c_1 + c_2 + 3 = 0$

$\Rightarrow c_1 = -c_2 - 3$

yes. there are infinitely many possibilities.

d) $c_1 + c_2 + 3 = 3$

$\Rightarrow c_1 + c_2 = 0$

$\Rightarrow c_1 = -c_2$

$4c_1 + 16c_2 + 3 = 15$

$-4c_2 + 16c_2 = 12$

$c_1 = -1 \qquad c_2 = 1$

Consider $c_1 e^{2x} + c_2 e^{x} = 0 \Rightarrow c_1 e^{x} + c_2 = 0$

$x=0$ $c_1 + c_2 = 0$ $x=1$ $c_1 e + c_2 = 0$

$\hookrightarrow -\frac{c_2}{c_1} = 1$ $-\frac{c_2}{c_1} = e$

Def: Linear Dependence/Independence

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

Def: Wronskian

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

Thm: Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous nth-order differential equation (1) on an interval I. Then the set of solutions is **linearly independent** on Iif and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

Note: The Wronskian is either always or never zero on an interval.

Ex: Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$.
$f_1(x) = \cos 2x$, $f_2(x) = 1$, $f_3(x) = \cos^2 x$

$\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos 2x - 1 - 2\cos^2 x = 0$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - fh$$

$$W(f_1, f_3, f_3) = \begin{vmatrix} \cos 2x & 1 & \cos^2 x \\ -2\sin 2x & 0 & \cancel{-2\cos x \sin x}\ -\sin 2x \\ -4\cos 2x & 0 & -2\cos 2x \end{vmatrix}$$

$$= -1\begin{vmatrix} -2\sin 2x & -\sin 2x \\ -4\cos 2x & -2\cos 2x \end{vmatrix} = -(4\sin 2x\cos 2x - 4\sin 2x\cos 2x)$$

$$= 0$$ dependent

Thm: Superposition Principle–Homogeneous Equations

Let $y_1, y_2, \ldots, y_k$, be solutions of the homogeneous nth-order differential equation (1) on an interval I. Then the linear combination $c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$, where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

Ex: $y_1 = e^{3x}$ and $y_2 = e^{-5x}$ are solutions to $y'' + 2y' - 15y = 0$

Then so is $y = c_1e^{3x} + c_2e^{-5x}$ for $c_1, c_2 \in \mathbb{R}$.

$y' = 3c_1e^{3x} - 5c_2e^{-5x}$

any multiple of e^{3x} is a solution. $y'' = 9c_1e^{3x} + 25c_2e^{-5x}$

Def: Fundamental Set of Solutions

Any set $y_1, y_2, \ldots, y_n$, of n of linearly independent solutions of the homogeneous nth-order differential equation (1) on an interval I is said to be a **fundamental set of solutions** on the interval.

Thm: General Solution–Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (1) on an interval I. Then the **general solution** of the equation on the interval is $y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$

Ex: Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

$x^2y'' + xy' + y = 0; \quad \cos(\ln x), \sin(\ln x), (0, \infty)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad y_1 \quad\quad\quad y_2$

Linear independence

$$W(y_1, y_2) = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ \frac{-\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix}$$

$$= \frac{\cos^2(\ln x)}{x} + \frac{\sin^2(\ln x)}{x} = \frac{1}{x}$$

$$\neq 0$$

linearly independent ✓

Solutions? $y_1 = \cos(\ln x)$, $y_1' = -\frac{\sin(\ln x)}{x}$

$$y_1'' = -\frac{\frac{\cos(\ln x)}{\cancel{x}}\cancel{x} - \sin(\ln x)}{x^2}$$

$$= \frac{\sin(\ln x) - \cos(\ln x)}{x^2}$$

check y_1:

$$\underbrace{\cancel{\sin(\ln x)} - \cancel{\cos(\ln x)}}_{x^2y''} \underbrace{- \cancel{\sin(\ln x)}}_{+xy'} + \underbrace{\cancel{\cos(\ln x)}}_{+y} = 0 \checkmark$$

General solution: $\boxed{y = c_1 \cos(\lambda_n x) + c_2 \sin(\lambda_n x)}$

Thm: Existence of a Fundamental Set

There exists a fundamental set of solutions for the homogenous linear nth-order differential equation (1) on an interval I.

Note that this solution has arbitrary parameters. (2-parameter family of solutions)

Def: Particular Solution

Any function y_p, free of arbitrary parameters, that satisfies (2) is said to be a **particular solution** of the equation.

Thm: General Solution–Nonhomogeneous Equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (2) on an interval I and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (1) on I. Then the **general solution** of the equation on the interval is $y = \boxed{c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)}+y_p(x)$, where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Thm: Superposition Principle–Nonhomogeneous Equations

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order differential equation (2) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \ldots, g_k$. Then $y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \ldots + y_{p_k}(x)$ is a particular solution of $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$ $= g_1(x) + g_2(x) + \ldots + g_k(x)$.

Ex: (a) Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$, respectively are particular solutions of
$y'' - 6y' + 5y = -9e^{2x}$ and
$y'' - 6y' + 5y = 5x^2 + 3x - 16.$

~~(b) Use part (a) to find particular solutions of~~
~~$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$ and~~
~~$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}$.~~

$$y_{p_1} = 3e^{2x} \Rightarrow y_{p_1}' = 6e^{2x}, \quad y_{p_1}'' = 12e^{2x}$$

$$12e^{2x} - 36e^{2x} + 15e^{2x} = -9e^{2x} \checkmark$$

$$y_{p_2} = x^2 + 3x \Rightarrow y_{p_2}' = 2x + 3, \quad y_{p_2}'' = 2$$

$$2 - 12x - 18 + 5x^2 + 15x$$

$$= 5x^2 + 3x - 16 \checkmark$$

Now write the particular solution to

$$y'' - 6y' + 5y = -9e^{2x} + 5x^2 + 3x - 16 \quad (*)$$

$$y_p = 3e^{2x} + x^2 + 3x$$

Note: the fundamental set of solutions to $y'' - 6y' + 5y = 0$ is $y_1 = e^x, y_2 = e^{5x}$

the general solution to $(*)$ is

$$y = \underbrace{c_1 e^x + c_2 e^{5x}}_{y_c} + \underbrace{3e^{2x} + x^2 + 3x}_{y_p}$$

$$y = y_c + y_p$$